MATH 579: Combinatorics

Exam 5 Solutions

1. Prove that $b_n = 3^n$ satisfies the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$.

   We compute $2b_{n-1} + 3b_{n-2} = 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} = 3^{n-2}(2 \cdot 3 + 3) = 3^{n-2}(9) = 3^n = b_n$.

2. Consider the recurrence given by $a_0 = 0, a_1 = 0, a_2 = 12, a_n = -3a_{n-1} + 4a_{n-3} + 18$ $(n \geq 3)$. Solve this using the methods of our packet.

   We start by considering the homogeneous recurrence relation $a_n = -3a_{n-1} + 4a_{n-3}$, with characteristic polynomial $x^3 + 3x^2 - 4 = (x-1)(x+2)^2$.

   Hence the general solution to the homogeneous problem is $a_n = A + B(-2)^n + Cn(-2)^n$.

   We turn now to the nonhomogeneous problem. The nonhomogeneous term is a polynomial of degree 0, but all such are already included in the nonhomogeneous case. Hence we instead guess $a_n = Dn$ as a solution. We get $Dn = -3D(n-1) + 4D(n-3) + 18$. The n terms all cancel, leaving $0 = 3D - 12D + 18$, so $D = 2$.

   The general nonhomogeneous solution is $a_n = 2n + A + B(-2)^n + Cn(-2)^n$.

   We now use our initial conditions: $0 = a_0 = 2 \cdot 0 + A + B(-2)^0 + C \cdot 0(-2)^0$, $0 = a_1 = 2 \cdot 1 + A + B(-2)^1 + C \cdot 1 \cdot (-2)^1$, $12 = a_2 = 2 \cdot 2 + A + B(-2)^2 + C \cdot 2 \cdot (-2)^2$. This gives system of equations $\{0 = A + B, 0 = 2 + A - 2B - 2C, 12 = 4 + A + 4B + 8C\}$. This has solution $A = 0, B = 0, C = 1$.

   Hence, the solution we seek is $a_n = 2n + n(-2)^n = n(2 + (-2)^n)$.

3. Consider the recurrence given by $a_0 = 0, a_1 = 0, a_2 = 12, a_n = -3a_{n-1} + 4a_{n-3} + 18$ $(n \geq 3)$. Solve this using generating functions.

   We set $A(x) = \sum_{n \geq 0} a_n x^n$. Multiplying our relation by x^n and summing over $n \geq 3$, we get $\sum_{n \geq 3} a_n x^n = -3 \sum_{n \geq 3} a_{n-1} x^n + 4 \sum_{n \geq 3} a_{n-3} x^n + 18 \sum_{n \geq 3} x^n$. Hence $A(x) - 0 - 0x - 12x^2 = -3x(A(x) - 0 - 0x) + 4x^3 A(x) + 18x^3 \frac{1}{1-x}$. We rearrange as $A(x)(1 + 3x - 4x^3) = 12x^2 + \frac{18x^3}{1-x} = \frac{12x^2 + 6x^3}{1-x}$. Hence $A(x) = \frac{12x^2 + 6x^3}{(1-x)(1+3x-4x^3)} = \frac{12x^2 + 6x^3}{(1-x)^2(1+2x)^2}$.

   We now have a partial fractions problem of $A(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+2x} + \frac{D}{(1+2x)^2}$, or $6x^2(2+x) = A(1-x)(1+2x)^2 + B(1+2x)^2 + C(1-x)^2(1+2x) + D(1-x)^2$. Taking $x = 1$, we get $18 = B(1+2)^2$, or $B = 2$. Taking $x = -\frac{1}{2}$, we get $6(\frac{1}{4})(\frac{3}{2}) = D(1 - \frac{-1}{2})^2$, or $D = 1$. Taking $x = 0$, we get $0 = A + B + C + D$, or $0 = A + C + 3$. Taking $x = -1$, we get $6(1)(1) = A(2)(-1)^2 + B(-1)^2 + C(2)^2(-1) + D(2)^2 = 2A + B - 4C + 4D = 2A - 4C + 6$, or $0 = A - 2C$. Solving, we get $A = -2, C = -1$.

   Hence, $A(x) = -2\sum_{n \geq 0} x^n + 2\sum_{n \geq 0}(n+1)x^n - \sum_{n \geq 0}(-2)^n x^n + \sum_{n \geq 0}(n+1)(-2)^n x^n = \sum_{n \geq 0}(-2 + 2(n+1) - (-2)^n + (n+1)(-2)^n)x^n$. Hence $a_n = -2 + 2(n+1) - (-2)^n + (n+1)(-2)^n = 2n + n(-2)^n = n(2 + (-2)^n)$.